REPRESENTATION OF SPECTRUMS OF C^*-ALGEBRAS OF BOUNDED FUNCTIONS IN TERMS OF FILTERS

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Abstract. Let X be a non-empty set and let $\mathcal{F}$ be any C^*-subalgebra of $\ell^\infty(X)$ containing the constant functions. We show that the spectrum of $\mathcal{F}$ can be considered as a space of certain filters determined by $\mathcal{F}$ on X. Furthermore, we show that these filters describe the topology of the spectrum of $\mathcal{F}$.

1. Introduction

It is well known that filters have a fundamental role in the study of algebraic properties of the Stone-Čech compactification βS of a discrete semigroup S. For any non-empty set X, let $\ell^\infty(X)$ denote the C^*-algebra of all bounded, complex-valued functions on X. The Stone-Čech compactification βX of a discrete topological space X can be viewed as the spectrum of $\ell^\infty(X)$. On the other hand, βX can also be viewed as the space of all ultrafilters on X (see [4] or [7]). The latter consideration of βS for a discrete semigroup S is the main tool in analyzing algebraic properties of βS in [7].

Any topological compactification of a completely regular topological space X or any semigroup compactification of a Hausdorff semitopological semigroup S is determined by the spectrum of some C^*-subalgebra $\mathcal{F}$ of $\ell^\infty(X)$ containing the constant functions. The purpose of the paper is to show that the spectrum of *any* C^*-subalgebra $\mathcal{F}$ of $\ell^\infty(X)$ containing the constant functions, where X is *any* non-empty set, can be considered as the space of all $\mathcal{F}$-ultrafilters on X. Independently of the C^*-algebra $\mathcal{F}$ in question, our approach has a number of similarities with the consideration of βX of a discrete topological space X as the space of all ultrafilters on X. For example, we obtain a bijective correspondence between non-empty, closed subsets of the spectrum of $\mathcal{F}$ and $\mathcal{F}$-filters on X. Considering the importance of filters in the study of algebraic properties of the Stone-Čech compactification βS of a discrete semigroup S and the similarities between our approach, we believe that the method presented in the paper can serve as a valuable tool in the

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study of semigroup compactifications and also of topological compactifications. In fact, this method was already used in [1] to study the smallest ideal and its closure in the $\mathcal{LUC}$-compactification of a topological group. Applications of our method to more general semigroup compactifications will appear in our upcoming paper.

Although the Stone-Čech compactification βX of a discrete topological space X is the most familiar compactification which may be considered as a space of filters, some other compactifications have also been studied in terms of filters. As far as we are aware, the representation of spectrums of C^*-algebras of bounded functions as spaces of filters developed in the paper is the most general one and, for many C^*-subalgebras of $\ell^\infty(X)$, the first one actually using filters. If X is a completely regular topological space, then the Stone-Čech compactification βX of X can be considered as the space of all z-ultrafilters on X (see [6] or [10]). If X is discrete, then our approach yields the usual representation of βX as the space of all ultrafilters on X, but for non-discrete spaces our approach gives a new representation of βX. In [8], the uniform compactification of a uniform space $(X,\mathcal{U})$ is considered as the space of all near ultrafilters on X. However, near ultrafilters need not be filters in the ordinary sense of the word, since they need not be closed under finite intersections. An approach using filters is given in [1]. The $\mathcal{WAP}$-compactification of a *discrete* semigroup is described in terms of filters in [3] and any semigroup compactification of a Hausdorff, semitopological semigroup S is considered as the space of certain equivalence classes of z-ultrafilters on S in [9].

We establish the representation of the spectrum of a C^*-subalgebra $\mathcal{F}$ of $\ell^\infty(X)$ containing the constant functions as the space of all $\mathcal{F}$-ultrafilters on X in Sections 4 and 5. We introduce these filters in Section 3 and describe some of their properties that we will use throughout the paper. These properties show that the approach presented in the paper is a natural extension of the consideration of βX of a discrete topological space X as the space of all ultrafilters on X. In Section 4, we also show that $\mathcal{F}$-filters describe the topology of the spectrum of $\mathcal{F}$ in a similar way as filters describe the topology of βX for a discrete topological space X. In Section 6, we describe some relationships between two C^*-subalgebras of $\ell^\infty(X)$ containing the constant functions. In Section 7, we establish a correspondence between $\mathcal{F}$-filters on X and closed, proper ideals of $\mathcal{F}$. In these sections, we assume no topological or algebraic structure on the set X. The last section is devoted to a study of $\mathcal{F}$-filters in the case that X is a Hausdorff, topological space and every member of $\mathcal{F}$ is continuous.

In the paper, we present a self-contained and elementary construction of the spectrum of a C^*-subalgebra $\mathcal{F}$ of $\ell^\infty(X)$ containing the constant functions.

Therefore, we wish to keep the prerequisites for the paper clear. The required results are Urysohn's Lemma, Dini's Theorem, and the fact that $|f| \in \mathcal{F}$ for every $f \in \mathcal{F}$ (see [2, pp. 89-90]).

2. Preliminaries

Throughout the paper, let X be a non-empty set and let $\mathcal{F}$ be a C^*-subalgebra of $\ell^\infty(X)$ containing the constant functions. We introduce shortly some notation and remind the reader of some definitions that we will use throughout the paper.

We denote the set of all positive integers by $\mathbb{N}$, that is, $\mathbb{N} = \{1, 2, 3, \ldots\}$. We denote the set of all subsets of X by $\mathcal{P}(X)$ and we define $\mathcal{P}^*(X) = \mathcal{P}(X)\backslash\{\emptyset\}$. A *filter* on X is a non-empty subset φ of $\mathcal{P}(X)$ with the following properties:

(i) If $A, B \in \varphi$, then $A \cap B \in \varphi$.
(ii) If $A \in \varphi$ and $A \subseteq B \subseteq X$, then $B \in \varphi$.
(iii) $\emptyset \notin \varphi$.

A *filter base* on X is a non-empty subset $\mathcal{B}$ of $\mathcal{P}(X)$ such that $\emptyset \notin \mathcal{B}$ and, for all sets $A, B \in \mathcal{B}$, there exists a set $C \in \mathcal{B}$ such that $C \subseteq A \cap B$. If $\mathcal{B}$ is a filter base on X, then the filter φ on X *generated* by $\mathcal{B}$ is

$$\varphi = \{A \subseteq X : \text{there exists some set } B \in \mathcal{B} \text{ such that } B \subseteq A\}.$$

Let φ be a filter on X. A subset $\mathcal{B}$ of $\mathcal{P}(X)$ is a *filter base* for φ if and only if $\mathcal{B} \subseteq \varphi$ and, for every set $A \in \varphi$, there exists a set $B \in \mathcal{B}$ such that $B \subseteq A$.

A real-valued function f on X is *positive* if and only if $f(x) \geq 0$ for every $x \in X$. If f and g are real-valued members of $\mathcal{F}$, then the functions $(f \vee g)$ and $(f \wedge g)$ in $\mathcal{F}$ are defined by $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$, respectively, for every $x \in X$.

Let (X, τ) be a (not necessarily Hausdorff) topological space. For every subset A of X, we denote by $\mathrm{int}_{(X,\tau)}(A)$ and $\mathrm{cl}_{(X,\tau)}(A)$ the interior and the closure of A in X, respectively, or simply by $\mathrm{int}_X(A)$ and $\mathrm{cl}_X(A)$ if τ is understood. We denote by $C(X)$ the C^*- subalgebra of $\ell^\infty(X)$ consisting of continuous members of $\ell^\infty(X)$. If X is a locally compact Hausdorff topological space, then the C^*-subalgebra $C_0(X)$ of $C(X)$ consists of those members of $C(X)$ which vanish at infinity.

3. $\mathcal{F}$-filters

In this section, we introduce $\mathcal{F}$-filters and $\mathcal{F}$-ultrafilters on X and we study some of their basic properties. In the next two sections, we show that the

spectrum of $\mathcal{F}$ can be considered as the space of all $\mathcal{F}$-ultrafilters on X and that $\mathcal{F}$-filters describe the topology of this space. For every $f \in \mathcal{F}$ and for every $r > 0$, we define

$$Z(f) = \{x \in X : f(x) = 0\} \quad \text{and} \quad X(f, r) = \{x \in X : |f(x)| \leq r\}.$$

Definition 1. An $\mathcal{F}$*-family* on X is a non-empty subset $\mathcal{A}$ of $\mathcal{P}^*(X)$ such that, for every set $A \in \mathcal{A}$ with $A \neq X$, there exist a set $B \in \mathcal{A}$ and a function $f \in \mathcal{F}$ such that $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. An $\mathcal{F}$*-filter* on X is a filter φ on X which is also an $\mathcal{F}$-family on X.

Of course, we may just as well assume that the function $f \in \mathcal{F}$ in the previous definition satisfies $f(B) = \{1\}$ and $f(X \setminus A) = \{0\}$. Also, since $|f| \in \mathcal{F}$ for every $f \in \mathcal{F}$, we may assume that f is real-valued and, if necessary, that $f(X) \subseteq [0, 1]$.

There exists at least one $\mathcal{F}$-filter on X, namely the filter $\varphi = \{X\}$. If $\mathcal{F}$ contains only constant functions, then $\{X\}$ is the only $\mathcal{F}$-filter on X. On the other hand, if $\mathcal{F} = \ell^\infty(X)$, then every filter φ on X is an $\mathcal{F}$-filter on X.

Let φ be an $\mathcal{F}$-filter on X and let $A \in \varphi$ with $A \neq X$. Pick a set $B \in \varphi$ and a function $f \in \mathcal{F}$ with $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Then $B \subseteq Z(f) \subseteq A$. Therefore, φ has a filter base consisting of zero sets (determined by $\mathcal{F}$) of X. However, not every zero set of X is in any $\mathcal{F}$-filter. For example, let $\mathcal{F} = C(\mathbb{R})$. Then $A = \{0\}$ is a zero set of $\mathbb{R}$ but there is no $\mathcal{F}$-filter φ on $\mathbb{R}$ such that $A \in \varphi$.

We shall apply the following remark frequently without any further notice.

Remark 2. Let $\mathcal{A}$ be a non-empty subset of $\mathcal{P}^*(X)$. Suppose that, for every set $A \in \mathcal{A}$ with $A \neq X$, there exist a set $B \in \mathcal{A}$, real numbers s and r with $s < r$, and a real-valued function $f \in \mathcal{F}$ such that $f(x) \leq s$ for every $x \in B$ and $f(x) \geq r$ for every $x \in X \setminus A$. Using the lattice operations $\vee$ and $\wedge$, we see that $\mathcal{A}$ is an $\mathcal{F}$-family on X.

Zorn's Lemma implies that every $\mathcal{F}$-filter on X is contained in some maximal (with respect to inclusion) $\mathcal{F}$-filter on X.

Definition 3. An $\mathcal{F}$*-ultrafilter* on X is an $\mathcal{F}$-filter on X which is not properly contained in any other $\mathcal{F}$-filter on X.

Note that if $\mathcal{F} = \ell^\infty(X)$, then a filter φ on X is an $\mathcal{F}$-ultrafilter if and only if φ is an ultrafilter on X. Also, the following simple fact about $\mathcal{F}$-ultrafilters is very useful: If p and q are $\mathcal{F}$-ultrafilters on X, then $p = q$ if and only if $p \subseteq q$.

Definition 4. Define

$$\mathcal{F}_0 = \{f \in \mathcal{F} : X(f, r) \neq \emptyset \text{ for every } r > 0\}.$$

For every non-empty subset A of X, define

$$\mathcal{Z}(A) = \{f \in \mathcal{F} : f(x) = 0 \text{ for every } x \in A\}.$$

The next lemma follows from Remark 2.

Lemma 5. If $\mathcal{F}' \subseteq \mathcal{F}_0$ is non-empty, then $\mathcal{A} = \{X(f, r) : f \in \mathcal{F}',\, r > 0\}$ is an $\mathcal{F}$-family on X.

We will use the following lemma and its corollaries a number of times in the paper. Recall that a non-empty subset $\mathcal{A}$ of $\mathcal{P}(X)$ has the *finite intersection property* if and only if $\bigcap_{k=1}^{n} A_k \neq \emptyset$ whenever $A_1, \dots, A_n \in \mathcal{A}$ for some $n \in \mathbb{N}$.

Lemma 6. If $\mathcal{A}$ is an $\mathcal{F}$-family on X such that $\mathcal{A}$ has the finite intersection property, then there exists an $\mathcal{F}$-ultrafilter p on X such that $\mathcal{A} \subseteq p$.

Proof. We sketch the proof briefly. Let φ be the smallest filter on X containing $\mathcal{A}$. Let $n \in \mathbb{N}$ and let $A_1, \dots A_n \in \mathcal{A}$ with $A_k \neq X$ for every $k \in \{1, \dots, n\}$. If $k \in \{1, \dots, n\}$, then there exist a set $B_k \in \mathcal{A}$ and a positive function $f_k \in \mathcal{F}$ with $f_k(B_k) = \{0\}$ and $f_k(X \setminus A_k) = \{1\}$. Put $B = \bigcap_{k=1}^{n} B_k$ and $f = \sum_{k=1}^{n} f_k$. Since $B \in \varphi$, $f \in \mathcal{F}$, $f(B) = \{0\}$, and $f(x) \geq 1$ for every $x \in X \setminus \bigcap_{k=1}^{n} A_k$, the filter φ is an $\mathcal{F}$-family on X. $\square$

The next two corollaries now follow from Lemma 5.

Corollary 7. Let φ be an $\mathcal{F}$-filter on X and let $f \in \mathcal{F}$. If $X(f, r) \cap B \neq \emptyset$ for all $r > 0$ and $B \in \varphi$, then there exists an $\mathcal{F}$-ultrafilter p on X with $\varphi \cup \{X(f, r) : r > 0\} \subseteq p$.

Corollary 8. Let φ be an $\mathcal{F}$-filter on X and let $A \subseteq X$. If $A \cap B \neq \emptyset$ for every $B \in \varphi$, then there exists an $\mathcal{F}$-ultrafilter p on X containing the family $\varphi \cup \{X(f, r) : f \in \mathcal{Z}(A),\, r > 0\}$.

If $\mathcal{F} = \ell^\infty(X)$, then we may take the functions in the next theorem to be characteristic functions of subsets of X. Except for statement (ii), the conclusion of the next theorem is the same as in [7, Theorem 3.6].

Theorem 9. If $\varphi \subseteq \mathcal{P}(X)$, then the following statements are equivalent:

(i) φ is an $\mathcal{F}$-ultrafilter on X.
(ii) φ is an $\mathcal{F}$-filter on X and, if $X(f, r) \notin \varphi$ for some $f \in \mathcal{F}$ and $r > 0$, then, for every real number t with $0 < t < r$, there exists a set $A \in \varphi$ with $X(f, t) \cap A = \emptyset$.

(iii) φ is a maximal subset of $\mathcal{P}(X)$ such that φ is an $\mathcal{F}$-family on X and φ has the finite intersection property.

(iv) φ is an $\mathcal{F}$-filter on X and, if $\bigcup_{k=1}^{n} A_k \in \varphi$ for some $n \in \mathbb{N}$ and for some subsets $A_1, \ldots, A_n$ of X, then there exists $k \in \{1, \ldots, n\}$ such that $X(f_k, r) \in \varphi$ for all $f \in \mathcal{Z}(A_k)$ and $r > 0$.

(v) φ is an $\mathcal{F}$-filter on X and, for every non-empty subset A of X with $A \neq X$, either $X(f, r) \in \varphi$ for all $f \in \mathcal{Z}(A)$ and $r > 0$, or $X(g, r) \in \varphi$ for all $g \in \mathcal{Z}(X \setminus A)$ and $r > 0$.

Proof. (i) $\Rightarrow$ (ii) This follows from Corollary 8 with $g = (|f| - t) \vee 0$. Note here that $g \in \mathcal{Z}(X(f, t))$ and $X(g, r - t) \subseteq X(f, r)$.

(ii) $\Rightarrow$ (iii) This follows from the definition of an $\mathcal{F}$-family.

(iii) $\Rightarrow$ (iv) Suppose that (iii) holds. Clearly, $X \in \varphi$, $\emptyset \notin \varphi$, and $B \in \varphi$ whenever $A \in \varphi$ and $A \subseteq B \subseteq X$. To see that φ is a filter, let $A, B \in \varphi$. Pick sets $C, D \in \varphi$ and functions $f, g \in \mathcal{F}$ such that $f(C) = f(D) = \{0\}$ and $f(X \setminus A) = g(X \setminus B) = \{1\}$. Since $C \cap D \subseteq X(|f| + |g|, r)$ for every $r > 0$, we have $X(|f| + |g|, r) \in \varphi$ for every $r > 0$ by Lemma 5. Since $X(|f| + |g|, 1/2) \subseteq A \cap B$, we have $A \cap B \in \varphi$, as required.

Suppose now that $\bigcup_{k=1}^{n} A_k \in \varphi$ for some $n \in \mathbb{N}$ and for some subsets $A_1, \ldots, A_n$ of X. Suppose also that, for every $k \in \{1, \ldots, n\}$, there exist $r_k > 0$ and $f_k \in \mathcal{Z}(A_k)$ such that $X(f_k, r_k) \notin \varphi$. If $k \in \{1, \ldots, n\}$, then $\mathcal{A} = \varphi \cup \{X(f_k, t) : t > 0\}$ is an $\mathcal{F}$-family on X by Lemma 5. By assumption, there exist a set $B_k \in \varphi$ and $t_k > 0$ such that $B_k \cap X(f_k, t_k) = \emptyset$. Put $B = \bigcap_{k=1}^{n} B_k \in \varphi$. Then $B \cap [\bigcup_{k=1}^{n} Z(f_k)] = \emptyset$, a contradiction.

(iv) $\Rightarrow$ (v) This is obvious.

(v) $\Rightarrow$ (i) Suppose that (v) holds. Suppose also that there exists an $\mathcal{F}$-filter ψ on X which properly contains φ. Pick some set $A \in \psi \setminus \varphi$. Pick a set $B \in \psi$ and a function $f \in \mathcal{F}$ such that $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Pick a set $C \in \psi$ and a function $g \in \mathcal{F}$ such that $g(C) = \{1\}$ and $g(X \setminus B) = \{0\}$. Since $X(f, 1/2) \subseteq A$, we have $X(f, 1/2) \notin \varphi$. Since $f \in \mathcal{Z}(B)$, we have $X(g, 1/2) \in \varphi$. But now $X(g, 1/2) \cap C = \emptyset$, a contradiction. $\square$

4. The topological space δX

In this section, we define a topology on the set of all $\mathcal{F}$-ultrafilters on X and establish some of the properties of the resulting space. In particular, we show that the resulting space is a compact Hausdorff space and that $\mathcal{F}$-filters describe the topology of this space.

Definition 10. Define $\delta X = \{p : p \text{ is an } \mathcal{F}\text{-ultrafilter on } X\}$. For every subset A of X, define $\widehat{A} = \{p \in \delta X : A \in p\}$. For every $\mathcal{F}$-filter φ on X, define $\widehat{\varphi} = \{p \in \delta X : \varphi \subseteq p\}$.

To be precise, we should include the C^*-algebra $\mathcal{F}$ in the notation above, such as $\delta_{\mathcal{F}}(X)$. Except in Section 6, we use only one C^*-algebra $\mathcal{F}$ in the same context, so the notation chosen above should not cause any misunderstandings.

Theorem 11. If φ and ψ are $\mathcal{F}$-filters on X, then the following statements hold:

(i) $\widehat{\varphi} = \bigcap_{A\in\varphi} \widehat{A}$.
(ii) $\varphi = \bigcap_{p\in\widehat{\varphi}} p$.
(iii) $\varphi \subseteq \psi$ if and only if $\widehat{\psi} \subseteq \widehat{\varphi}$.
(iv) $\varphi = \psi$ if and only if $\widehat{\varphi} = \widehat{\psi}$.

Proof. (i) This is obvious.

(ii) The inclusion $\varphi \subseteq \bigcap_{p\in\widehat{\varphi}} p$ is obvious, and so we need only to verify the reverse inclusion. Suppose that $A \subseteq X$ satisfies $A \notin \varphi$. By Corollary 8, there exists an element $p \in \widehat{\varphi}$ such that $\{X(f, r) : f \in \mathcal{Z}(X \setminus A),\ r > 0\} \subseteq p$. Now, it is enough to show that $A \notin p$. Suppose that $A \in p$. Pick a set $B \in p$ and a function $f \in \mathcal{F}$ such that $f(B) = \{1\}$ and $f(X \setminus A) = \{0\}$. Since $f \in \mathcal{Z}(X \setminus A)$, we have $X(f, 1/2) \in p$. But now $B \cap X(f, 1/2) = \emptyset$, a contradiction.

(iii) Necessity is obvious and sufficiency follows from statement (ii).

(iv) This follows from statement (iii). $\square$

The family $\{\widehat{A} : A \subseteq X\}$ is a base for a topology on δX. We define the topology of δX to be the topology which has the family $\{\widehat{A} : A \subseteq X\}$ as its base. In particular, $\{\widehat{A} : A \in p\}$ is the neighborhood base of a point $p \in \delta X$. If $Y \subseteq \delta X$, then we denote $\mathrm{cl}_{\delta X}(Y)$ by $\overline{Y}$ with one exception: If $A \subseteq X$, then we use $\mathrm{cl}_{\delta X}(\widehat{A})$ instead of the cumbersome notation $\overline{\widehat{A}}$.

We denote by $\tau(\mathcal{F})$ the weakest topology τ on X such that every function f in $\mathcal{F}$ is continuous with respect to τ. For every subset A of X, we denote $\mathrm{int}_{(X,\tau(\mathcal{F}))}(A)$ by A°. For every element $x \in X$, we denote by $\mathcal{N}_{\mathcal{F}}(x)$ the neighborhood filter of x in $(X, \tau(\mathcal{F}))$.

We shall apply the following remark frequently without any further notice.

Remark 12. Let φ be an $\mathcal{F}$-filter on X and let $A \in \varphi$ with $A \neq X$. Pick $B \in \varphi$ and $f \in \mathcal{F}$ such that $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Then

$B \subseteq \{x \in X : |f(x)| < 1\} \subseteq A$. Therefore, if C is any subset of X, then $C \in \varphi$ if and only if $C^\circ \in \varphi$.

Theorem 13. If $x \in X$, then the family

$$\mathcal{A}_x = \{X(f,r) : f \in \mathcal{F},\ f(x) = 0,\ \text{and } r > 0\}$$

is a filter base on X. The filter on X generated by $\mathcal{A}_x$ is the filter $\mathcal{N}_{\mathcal{F}}(x)$ and it is an $\mathcal{F}$-ultrafilter.

Proof. If $f, g \in \mathcal{F}$ and $r > 0$, then $X(|f| + |g|, r) \subseteq X(f,r) \cap X(g,r)$. Therefore, $\mathcal{A}_x$ is a filter base on X. Clearly, $\mathcal{A}_x$ generates the filter $\mathcal{N}_{\mathcal{F}}(x)$, and so $\mathcal{N}_{\mathcal{F}}(x)$ is an $\mathcal{F}$-filter on X by Lemma 5. Then $\mathcal{N}_{\mathcal{F}}(x)$ is an $\mathcal{F}$-ultrafilter on X by Theorem 9 (iv). □

The following definition is reasonable by the previous theorem.

Definition 14. The *evaluation mapping* $e : X \to \delta X$ is defined by $e(x) = \mathcal{N}_{\mathcal{F}}(x)$.

If $A \subseteq X$ and $x \in X$, then $e(x) \in \widehat{A}$ if and only if $x \in A^\circ$. Next, let $A, B \subseteq X$. In general, $\widehat{B} \cap e(A) = \emptyset$ does not imply $B \cap A = \emptyset$. However, this implication holds if B is a $\tau(\mathcal{F})$-open subset of X. We apply this fact repeatedly in what follows.

We collect some properties of the space δX in the following lemmas.

Lemma 15. Let $A \subseteq X$ and let $p \in \delta X$. The following statements are equivalent:

(i) $p \in \overline{e(A)}$.
(ii) $A \cap B \neq \emptyset$ for every $B \in p$.
(iii) $X(f,r) \in p$ for every $f \in \mathcal{Z}(A)$ and for every $r > 0$.

In particular, $p \in \overline{e(A)}$ for every $A \in p$.

Proof. (i) $\Rightarrow$ (ii) If $A \cap B = \emptyset$ for some $B \in p$, then $A \cap B^\circ = \emptyset$, and so $e(A) \cap \widehat{B^\circ} = \emptyset$. Since $B^\circ \in p$, we have $p \notin \overline{e(A)}$.

(ii) $\Rightarrow$ (iii) This follows from Corollary 8.

(iii) $\Rightarrow$ (i) Suppose that $p \notin \overline{e(A)}$. Pick a $\tau(\mathcal{F})$-open subset B of X such that $B \in p$ and $\widehat{B} \cap e(A) = \emptyset$. Then $B \cap A = \emptyset$. Pick a set $C \in p$ and a function $f \in \mathcal{F}$ such that $f(C) = \{1\}$ and $f(X \setminus B) = \{0\}$. Then $f \in \mathcal{Z}(A)$. Since $X(f, 1/2) \cap C = \emptyset$, we have $X(f, 1/2) \notin p$. □

Lemma 16. If $A, B \subseteq X$, then the following statements hold:

(i) $\widehat{X \setminus A} = \delta X \setminus \overline{e(A)}$.
(ii) If A is a $\tau(\mathcal{F})$-open subset of X, then $\overline{e(A)} = \mathrm{cl}_{\delta X}(\widehat{A})$.
(iii) $\widehat{A} = \widehat{B}$ if and only if $A^\circ = B^\circ$.
(iv) $\widehat{A} = \emptyset$ if and only if $A^\circ = \emptyset$.
(v) $\widehat{A} = \delta X$ if and only if $A = X$.

Proof. (i) Let $p \in \widehat{X \setminus A}$. Since $\widehat{X \setminus A} \cap e(A) = \emptyset$, we have $p \notin \overline{e(A)}$. On the other hand, if $p \in \delta X \setminus \overline{e(A)}$, then there exists a $\tau(\mathcal{F})$-open subset C of X such that $C \in p$ and $\widehat{C} \cap e(A) = \emptyset$. Then $C \cap A = \emptyset$, and so $X \setminus A \in p$, as required.

(ii) The inclusion $\mathrm{cl}_{\delta X}(\widehat{A}) \subseteq \overline{e(A)}$ holds for any subset A of X and follows from statement (i). Suppose now that A is a $\tau(\mathcal{F})$-open subset of X and let $p \in \overline{e(A)}$. If $B \in p$, then $\widehat{B} \cap e(A) \neq \emptyset$, so $B^\circ \cap A \neq \emptyset$, and so $\widehat{B} \cap \widehat{A} \neq \emptyset$. Therefore, $p \in \mathrm{cl}_{\delta X}(\widehat{A})$.

Statement (iii) follows from Remark 12. Then (iv) and (v) follow from (iii). □

Lemma 17. If $A, B \subseteq X$, then $X(f,r) \cap X(g,r) \neq \emptyset$ for all $f \in \mathcal{Z}(A)$, $g \in \mathcal{Z}(B)$, and $r > 0$ if and only if $\overline{e(A)} \cap \overline{e(B)} \neq \emptyset$.

Proof. Necessity follows from Lemma 15. Suppose that $X(f,r) \cap X(g,r) \neq \emptyset$ for all $f \in \mathcal{Z}(A)$, $g \in \mathcal{Z}(B)$, and $r > 0$. Put

$$\mathcal{A} = \{X(h,r) : h \in \mathcal{Z}(A) \cup \mathcal{Z}(B),\ r > 0\}.$$

Then $\mathcal{A}$ is an $\mathcal{F}$-family on X by Lemma 5. If $f_1, \dots, f_n \in \mathcal{Z}(A)$ and $g_1, \dots, g_m \in \mathcal{Z}(B)$ for some $n, m \in \mathbb{N}$, then $f := \sum_{k=1}^n |f_k| \in \mathcal{Z}(A)$, $g := \sum_{k=1}^m |g_k| \in \mathcal{Z}(B)$. If $r > 0$, then

$$X(f,r) \cap X(g,r) \subseteq \Big(\bigcap_{k=1}^n X(f_k, r)\Big) \cap \Big(\bigcap_{k=1}^m X(g_k, r)\Big),$$

and so $\mathcal{A}$ has the finite intersection property. By Lemma 6, there exists an element $p \in \delta X$ such that $\mathcal{A} \subseteq p$. Then $p \in \overline{e(A)} \cap \overline{e(B)}$ by Lemma 15. □

Now, we are ready to prove first of the main theorems of this section.

Theorem 18. The space δX is a compact Hausdorff space and $e(X)$ is dense in δX.

Proof. The density of $e(X)$ in δX follows from Lemma 16 (iv). To see that δX is Hausdorff, let p and q be distinct points of δX. Pick some set $A \in p \setminus q$. Pick a set $B \in p$ and a function $f \in \mathcal{F}$ such that $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Since $X(f, 1/2) \notin q$, there exists a set $C \in q$ such that

$X(f,1/3) \cap C = \emptyset$ by Theorem 9 (ii). Then $B \cap C = \emptyset$, and so $\widehat{B}$ and $\widehat{C}$ are disjoint neighborhoods of p and q, respectively.

Let us show that δX is compact. Lemma 16 (i) implies that the family $\mathcal{B} = \{\overline{e(A)} : A \subseteq X\}$ is a base for the closed sets of δX. Suppose that a subset $\mathcal{C}$ of $\mathcal{B}$ has the finite intersection property. To show that δX is compact, it is enough to show that $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$. Put $\mathcal{A}' = \{A \subseteq X : \overline{e(A)} \in \mathcal{C}\}$ and, then, put $\mathcal{A} = \{X(f,r) : A \in \mathcal{A}',\ f \in \mathcal{Z}(A),\ r > 0\}$. Now, $\mathcal{A}$ is an $\mathcal{F}$-family on X by Lemma 5 and $\mathcal{A}$ has the finite intersection property by Lemma 17. By Lemma 6, there exists an element $p \in \delta X$ such that $\mathcal{A} \subseteq p$. Then $p \in \overline{e(A)}$ for every $A \in \mathcal{A}'$ by Lemma 15, and so $p \in \bigcap_{C \in \mathcal{C}} C$, as required. □

We finish this section by showing that $\mathcal{F}$-filters describe the topology of δX. As in the Stone-Čech compactification of a discrete topological space, we have two interpretations for the closure of an $\mathcal{F}$-filter in δX, namely $\widehat{\varphi}$ and the following.

Definition 19. Define $\overline{\varphi} = \bigcap_{A \in \varphi} \overline{e(A)}$ for every $\mathcal{F}$-filter φ on X.

Note that $\overline{\varphi}$ is a non-empty, closed subset of δX.

Theorem 20. If φ is an $\mathcal{F}$-filter on X, then $\widehat{\varphi} = \overline{\varphi}$.

Proof. The inclusion $\widehat{\varphi} \subseteq \overline{\varphi}$ follows from Lemma 15. To prove the reverse inclusion, let $p \in \overline{\varphi}$ and let $A \in \varphi$ with $A \neq X$. Pick a set $B \in \varphi$ and a function $f \in \mathcal{F}$ such that $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Since $p \in \overline{e(B)}$, we have $X(f,1/2) \in p$ by Lemma 15, and so $A \in p$. Therefore, $\varphi \subseteq p$, as required. □

Theorem 21. If C is a non-empty, closed subset of δX, then there exists a unique $\mathcal{F}$-filter φ on X such that $\widehat{\varphi} = C$.

Proof. Let C be a non-empty, closed subset of δX. Put $\varphi = \bigcap_{p \in C} p$. Clearly, φ is a filter on X. Let us show that φ is an $\mathcal{F}$-family on X, hence, an $\mathcal{F}$-filter on X. Let $A \in \varphi$. If $p \in C$, then $A \in p$, and so there exist a set $B_p \in p$ and a function $f_p \in \mathcal{F}$ such that $f(X) \subseteq [0,1]$, $f_p(B_p) = \{0\}$, and $f_p(X \setminus A) = \{1\}$. Now, $\{\widehat{B}_p : p \in C\}$ is an open cover of C, and so there exist some $n \in \mathbb{N}$ and points $p_1, \dots, p_n \in C$ such that $C \subseteq \bigcup_{k=1}^{n} \widehat{B}_{p_k}$. Put $f = \sum_{k=1}^{n} f_{p_k}$ and $B = \bigcup_{k=1}^{n} B_{p_k}$. Then $B \in \varphi$ by Theorem 11. Since $f(x) \leq n-1$ for every $x \in B$ and $f(x) = n$ for every $x \in X \setminus A$, the filter φ is an $\mathcal{F}$-family on X.

Let us verify the equality $\widehat{\varphi} = C$. The inclusion $C \subseteq \widehat{\varphi}$ is obvious, so suppose that $q \in \delta X \setminus C$. Then there exists a $\tau(\mathcal{F})$-open subset A of X such that $A \in q$ and $\widehat{A} \cap C = \emptyset$. For every $p \in C$, pick a $\tau(\mathcal{F})$-open subset B_p of X such that $B_p \in p$ and $\widehat{A} \cap \widehat{B}_p = \emptyset$. Then $A \cap B_p = \emptyset$ for every $p \in C$. As above,

there exist $n \in \mathbb{N}$ and points $p_1, \ldots, p_n \in C$ such that $B := \bigcup_{k=1}^{n} B_{p_k} \in \varphi$. Since $A \cap B = \emptyset$, we have $q \notin \widehat{\varphi}$, as required.

Finally, uniqueness of φ follows from Theorem 11 (iv). $\square$

5. Continuous functions on δX

This section is devoted to a study of continuous functions on the space δX. We show that the C^*-algebras $\mathcal{F}$ and $C(\delta X)$ are isometrically $*$-isomorphic, that is, we show that δX is the spectrum of $\mathcal{F}$. Furthermore, we show that every dense image of X in a compact Hausdorff space is determined up to homeomorphism by some C^*-subalgebra of $\ell^\infty(X)$ containing the constant functions.

We leave the proof of the following simple lemma to the reader.

Lemma 22. If $p \in \delta X$, $g \in C(\delta X)$, and $r > 0$, then

$$\{x \in X : |g(p) - g(e(x))| \leq r\} \in p.$$

Theorem 23. If $f \in \mathcal{F}$, then there exists a unique function $\widehat{f} \in C(\delta X)$ with $f = \widehat{f} \circ e$.

Proof. Let $p \in \delta X$ and define

$$C = \bigcap_{A \in p} \mathrm{cl}_{\mathbb{C}}(f(A)).$$

Since f is bounded and p has the finite intersection property, the set C is a non-empty subset of $\mathbb{C}$. Choosing any element $\widehat{f}(p) \in C$, we obtain a function $\widehat{f} : \delta X \to \mathbb{C}$.

Next, let us show that if $p = \mathcal{N}_{\mathcal{F}}(x)$ for some $x \in X$, then $C = \{f(x)\}$. This will establish the equality $f = \widehat{f} \circ e$. Clearly, $f(x) \in C$. To see that $f(x)$ is the only member of C, let $y \in \mathbb{C}$ with $y \neq f(x)$. Pick $r > 0$ such that $y \notin U := \{z \in \mathbb{C} : |f(x) - z| \leq r\}$. Then $f^{-1}(U) \in \mathcal{N}_{\mathcal{F}}(x)$. Since $y \notin \mathrm{cl}_{\mathbb{C}}(f(f^{-1}(U)))$, we have $y \notin C$, as required.

Finally, let us show that $\widehat{f}$ is continuous. This will also prove the uniqueness of $\widehat{f}$, since $e(X)$ is dense in δX. Let $p \in \delta X$ and put $g = f - \widehat{f}(p)$. We claim that $X(g, r) \in p$ for every $r > 0$. By Corollary 7, it is enough to show that $X(g, r) \cap B \neq \emptyset$ for every $B \in p$ and for every $r > 0$. So, let $B \in p$ and $r > 0$ be given. Since $\widehat{f}(p) \in \mathrm{cl}_{\mathbb{C}}(f(B))$, there exists a point $x \in B$ such that $|g(x)| = |f(x) - \widehat{f}(p)| \leq r$, and so $x \in X(g, r) \cap B$, as required. To finish the proof, let $r > 0$. If $q \in \widehat{X(g, r)}$, then $\widehat{f}(q) \in \mathrm{cl}_{\mathbb{C}}(f(X(g, r)))$, and so $|\widehat{f}(q) - \widehat{f}(p)| \leq r$. Therefore, $\widehat{f}$ is continuous at p. $\square$

We defined the function $\widehat{f}$ by choosing any element from the set C. Since $\widehat{f}$ is unique, the set C must be a singleton for every $p \in \delta X$. Although the evaluation mapping e need not be injective, we call the function $\widehat{f}$ an *extension* of f to δX.

Following theorem can be deduced using the Stone-Weierstrass Theorem. However, we present the following proof using only properties of $\mathcal{F}$-filters instead of the Stone-Weierstrass Theorem.

Theorem 24. The mapping $\Gamma : \mathcal{F} \to C(\delta X)$ defined by $\Gamma(f) = \widehat{f}$ is an isometric $*$-isomorphism.

Proof. Using the density of $e(X)$ in δX and the equality $f = \widehat{f} \circ e$, it is easy to verify that Γ is an isometric $*$-homomorphism and we leave the details to the reader. To see that Γ is surjective, it is enough to show that for every positive function $g \in C(\delta X)$ with $\|g\| = 1$ and for every $r > 0$, there exists a function $f \in \mathcal{F}$ such that $\|\widehat{f} - g\| \leq r$. So, let $g \in C(\delta X)$ be positive with $\|g\| = 1$ and let $r > 0$. Pick $n \in \mathbb{N}$ such that $1/n \leq r/3$. For every $k \in \{1, \dots, n\}$, define the following subsets of $[0, 1]$, X, and δX, respectively:

$$
\begin{aligned}
I_k &= [\frac{k-1}{n}, \frac{k}{n}], \\
A_k &= \{x \in X : \frac{k-2}{n} < g(e(x)) < \frac{k+1}{n}\}, \\
C_k &= g^{-1}(I_k).
\end{aligned}
$$

Note that $A_k \cap A_j = \emptyset$ whenever $k, j \in \{1, \dots, n\}$ and $k + 3 \leq j$.

Let $k \in \{1, \dots, n\}$. If $p \in C_k$, then $A_k \in p$ by Lemma 22, and so there exist a set $B_p \in p$ and a positive function $f_p \in \mathcal{F}$ such that $f_p(B_p) = \{k/n\}$, $f_p(X \setminus A_k) = \{0\}$, and $f_p(X) \subseteq [0, k/n]$. Pick $p_1, \dots, p_n \in C_k$ for some $n \in \mathbb{N}$ such that $C_k \subseteq \bigcup_{j=1}^{n} \widehat{B}_{p_j}$ and put $f_k = f_{p_1} \vee \ldots \vee f_{p_n}$. Note that $f_k(X \setminus A_k) = \{0\}$ and $f_k(x) = k/n$ for every $x \in X$ with $e(x) \in C_k$.

Put $f = f_1 \vee \ldots \vee f_n$. We claim that $\|\widehat{f} - g\| \leq r$. To prove this, it is enough to show that $|f(x) - g(e(x))| \leq r$ for every $x \in X$. So, let $x \in X$. If $g(e(x)) \geq (n-3)/n$, then $e(x) \in C_k$ for some $k \in \mathbb{N}$ with $n - 2 \leq k \leq n$, and so $f(x) \geq (n-2)/n$. Therefore, $|f(x) - g(e(x))| \leq r$. On the other hand, if $g(e(x)) < (n-3)/n$, then there exists an element $k \in \{1, \dots, n-3\}$ such that $(k-1)/n \leq g(e(x)) < k/n$. Now, $x \in A_k$ and $e(x) \in C_k$, and so $f(x) \geq k/n$. Since $A_k \cap A_j = \emptyset$ for every $j \in \{1, \dots, n\}$ with $j \geq k + 3$, we have $f_j(x) = 0$ for every $k + 3 \leq j \leq n$, and so $f(x) \leq (k+2)/n$. Therefore, $|f(x) - g(e(x))| \leq 3/n \leq r$, thus finishing the proof. $\square$

Next, we show that $\mathcal{F}$-filters describe all dense images of X in compact Hausdorff spaces. Precise statement and details follow.

Theorem 25. Suppose that Y is a compact Hausdorff space and $\varepsilon : X \to Y$ is a function such that $\varepsilon(X)$ is dense in Y. The following statements hold:

(i) The set $\mathcal{F} = \{h \circ \varepsilon : h \in C(Y)\}$ is a C^*-subalgebra of $\ell^\infty(X)$ containing the constant functions.
(ii) $\mathcal{F}$ is isometrically isomorphic with $C(Y)$.
(iii) There exists a homeomorphism $F : \delta X \to Y$ such that $F \circ e = \varepsilon$.

Proof. We only prove statement (iii) and leave the easy verifications of (i) and (ii) to the reader. If $p \in \delta X$, then

$$C = \bigcap_{A \in p} \mathrm{cl}_Y(\varepsilon(A))$$

is a non-empty subset of Y. Suppose that $x, y \in C$ with $x \neq y$. By Urysohn's Lemma, there exists a real-valued function $h \in C(Y)$ with $h(x) = 0$ and $h(y) = 1$. Put $f = h \circ \varepsilon$ and $A = \{x \in X : \widehat{f}(p) - 1/3 \leq f(x) \leq \widehat{f}(p) + 1/3\}$. Now, $A \in p$ by Lemma 22, so $x, y \in \mathrm{cl}_Y(\varepsilon(A))$, and so $h(x), h(y) \in \mathrm{cl}_{\mathbb{R}}(f(A))$. Therefore, $|h(x) - h(y)| \leq 2/3$, a contradiction.

Since C is a singleton, we obtain a function $F : \delta X \to Y$. Clearly, $F \circ e = \varepsilon$. Since $e(X)$ and $\varepsilon(X)$ are dense in δX and Y, respectively, we need only to show that F is injective and continuous to finish the proof.

Let $p, q \in \delta X$ with $p \neq q$. By Urysohn's Lemma, there exists a real-valued function $g \in C(\delta X)$ such that $g(p) = 0$ and $g(q) = 1$. Put $f = g \circ e$. Then $f \in \mathcal{F}$ by Theorem 24. Put $A = \{x \in X : f(x) \leq 1/3\}$ and $B = \{x \in X : f(x) \geq 2/3\}$. Then $A \in p$ and $B \in q$ by Lemma 22, and so $F(p) \in \mathrm{cl}_Y(\varepsilon(A))$ and $F(q) \in \mathrm{cl}_Y(\varepsilon(B))$. By statement (ii), there exists a function $h \in C(Y)$ such that $f = h \circ \varepsilon$. Then $h(F(p)) \in \mathrm{cl}_{\mathbb{R}}(f(A))$ and $h(F(q)) \in \mathrm{cl}_{\mathbb{R}}(f(B))$, and so $F(p) \neq F(q)$, as required.

To finish the proof, let $p \in \delta X$ and let U be an open neighborhood of $F(p)$ in Y with $U \neq Y$. Again, there exists a real-valued function $h \in C(Y)$ such that $h(\widehat{f}(p)) = 0$ and $h(Y \backslash U) = \{1\}$. Put $f = h \circ \varepsilon$. Using the continuity of h as above, we obtain $\widehat{f}(p) = 0$, and so $B = \{x \in X : -1/2 \leq f(x) \leq 1/2\} \in p$ by Lemma 22. If $q \in \widehat{B}$, then $h(F(q)) \in [-1/2, 1/2]$, and so $F(q) \in U$. Therefore, F is continuous at p. $\square$

6. Some relationships between C^*-subalgebras of $\ell^\infty(X)$

Throughout this section, we assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are C^*-subalgebras of $\ell^\infty(X)$ containing the constant functions. We denote by $\delta_1 X$, and $\delta_2 X$ the spaces of $\mathcal{F}_1$-ultrafilters and $\mathcal{F}_2$-ultrafilters on X, respectively. Also, we denote by e_1 and e_2 the evaluation mappings from X to $\delta_1 X$ and $\delta_2 X$,

respectively. If $A \subseteq X$, then the notation $\widehat{A}$ is ambiguous. However, it should be clear from the context whether we consider $\widehat{A}$ as a subset of $\delta_1 X$ or $\delta_2 X$. If $f \in \mathcal{F}_1 \cap \mathcal{F}_2$, then f extends to both $\delta_1 X$ and $\delta_2 X$. We denote these extension by f^{δ_1} and f^{δ_2}, respectively.

Theorem 26. The inclusion $\mathcal{F}_1 \subseteq \mathcal{F}_2$ holds if and only if there exists a continuous, surjective mapping $F : \delta_2 X \to \delta_1 X$ such that $e_1 = F \circ e_2$.

Proof. Suppose first that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Let $p \in \delta_2 X$ and put

$$C = \bigcap_{A \in p} \mathrm{cl}_{\delta_1 X}(e_1(A)).$$

Similar arguments as in the proof of Theorem 25 apply to show that C is a singleton, and so we obtain a function $F : \delta_2 X \to \delta_1 X$. Clearly, $e_1 = F \circ e_2$. Also, arguing as in the last part of the proof of Theorem 25 we see that F is continuous. Therefore, we need only to show that F is surjective.

If $q \in \delta_1 X$, then q is an $\mathcal{F}_2$-filter on X. Pick any $p \in \delta_2 X$ such that $q \subseteq p$. Let $A \in q$. Since $\delta_1 X$ is a regular topological space, there exists a $\tau(\mathcal{F}_1)$-open subset B of X such that $B \in q$ and $\mathrm{cl}_{\delta_1 X}(\widehat{B}) \subseteq \widehat{A}$. Now, $B \in p$, so $F(p) \in \mathrm{cl}_{\delta_1 X}(\widehat{B})$ by Lemma 16 (ii), and so $A \in F(p)$. Therefore, $q \subseteq F(p)$, and so $q = F(p)$, as required.

Suppose now that there exists a mapping $F : \delta_2 X \to \delta_1 X$ as above. Let $f \in \mathcal{F}_1$. By Theorem 23, there exists a function $g \in C(\delta_1 X)$ with $f = g \circ e_1$, and so $f = (g \circ F) \circ e_2$. Since $g \circ F \in C(\delta_2 X)$, we have $f \in \mathcal{F}_2$ by Corollary 24. $\square$

For the proof of the next theorem, recall the definition of $\widehat{f}$ from the proof of Theorem 23.

Theorem 27. Suppose that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and let $F : \delta_2 X \to \delta_1 X$ be as in Theorem 26. If $p \in \delta_2 X$ and $q \in \delta_1 X$, then the following statements are equivalent:

(i) $q \subseteq p$.
(ii) $F(p) = q$.
(iii) $f^{\delta_2}(p) = f^{\delta_1}(q)$ for every $f \in \mathcal{F}_1$.

Proof. (i) $\Rightarrow$ (ii) This was already proved in the proof of Theorem 26.

(ii) $\Rightarrow$ (iii) Suppose that $F(p) = q$. Let $f \in \mathcal{F}_1$. Since $e_1 = F \circ e_2$, the functions f^{δ_2} and $f^{\delta_1} \circ F$ agree on $e_2(X)$. Therefore, $f^{\delta_2} = f^{\delta_1} \circ F$, and so $f^{\delta_2}(p) = f^{\delta_1}(q)$.

(iii) $\Rightarrow$ (i) Suppose that q is not contained in p. Pick some set $A \in q \setminus p$. Pick a set $B \in q$ and a positive function $f \in \mathcal{F}_1$ such that $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Then $f^{\delta_1}(q) = 0$. Since $X(f, 1/2) \notin p$, there exists a set $C \in p$ such that $X(f, 1/3) \cap C = \emptyset$ by Theorem 9. Then $f^{\delta_2}(p) \geq 1/3$, thus finishing the proof. $\square$

Define two closed equivalence relations $\sim$ and $\approx$ on $\delta_2 X$ as follows: $p \sim q$ if and only if $F(p) = F(q)$, and $p \approx q$ if and only if $f^{\delta_2}(p) = f^{\delta_2}(q)$ for every $f \in \mathcal{F}_1$. Theorem 27 shows that these relations are identical. Since F is a quotient mapping (see [11, pp. 60-61]), we obtain the following statement.

Corollary 28. If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then the quotient space $\delta_2 X/\approx$ is homeomorphic with $\delta_1 X$.

7. $\mathcal{F}$-FILTERS AND IDEALS OF $\mathcal{F}$

In this section, we establish a correspondence between $\mathcal{F}$-filters on X and ideals of $\mathcal{F}$. Roughly speaking, we show how the ideals of $\mathcal{F}$ can be used to generate $\mathcal{F}$-filters on X. We apply the following convention for the rest of the paper: *By an ideal of $\mathcal{F}$, we always mean a closed, proper ideal of $\mathcal{F}$.*

The next lemma follows from [5, (1.23) Proposition]. Since the proof of the cited proposition relies on the spectrums of single elements of C^*-algebras, we present the following short proof using only basic properties of Banach algebras.

Lemma 29. If $f \in \mathcal{F} \setminus \mathcal{F}_0$, then $1/f \in \mathcal{F}$.

Proof. Suppose first that $f \in \mathcal{F} \setminus \mathcal{F}_0$ is positive. Pick $r > 0$ such that $r \leq f(x)$ for every $x \in X$. Then

$$0 < \frac{r}{\|f\|} \leq \frac{f(x)}{\|f\|} \leq 1$$

for every $x \in X$. Put $g = f/\|f\|$. Then $\|1-g\| < 1$ by the inequalities above, and so g is invertible in $\mathcal{F}$ (see [5, (1.3) Lemma]). Therefore, $1/f \in \mathcal{F}$.

If $f \in \mathcal{F} \setminus \mathcal{F}_0$ is any function, then $|f|^2 \in \mathcal{F} \setminus \mathcal{F}_0$ is positive. The equality $1/f = \overline{f}/|f|^2$ and the first part of the proof imply that $1/f \in \mathcal{F}$. $\square$

Corollary 30. If I is an ideal of $\mathcal{F}$, then $I \subseteq \mathcal{F}_0$.

Definition 31. For every ideal I of $\mathcal{F}$, define

$$\mathcal{B}(I) = \{X(f, r) : f \in I, r > 0\}.$$

Theorem 32. If φ is an $\mathcal{F}$-filter on X, then there exists an ideal I of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$. Conversely, if I is an ideal of $\mathcal{F}$, then $\mathcal{B}(I)$ is a filter base on X and the filter φ on X generated by $\mathcal{B}(I)$ is an $\mathcal{F}$-filter.

Proof. Suppose first that φ is an $\mathcal{F}$-filter on X. Put

$$I = \{f \in \mathcal{F} : X(f,r) \in \varphi \text{ for every } r > 0\}.$$

Clearly, $0 \in I$. Let $f_1, f_2 \in I$, let $h \in \mathcal{F}$ with $h \neq 0$, let $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, let (g_n) be a sequence in I which converges to some $g \in \mathcal{F}$, and let $r > 0$. The inclusions

$$\begin{aligned} X(f_1, r/2) \cap X(f_2, r/2) &\subseteq X(f_1 - f_2, r), \\ X(f_1, r/\|h\|) &\subseteq X(f_1 h, r), \\ X(f_1, r/|\alpha|) &\subseteq X(\alpha f_1, r), \\ X(g_n, r/2) &\subseteq X(g, r), \end{aligned}$$

where the last one holds if $\|g_n - g\| \leq r/2$, imply that I is an ideal of $\mathcal{F}$.

We claim that $\mathcal{B}(I)$ is a filter base for φ. Clearly, $\mathcal{B}(I) \subseteq \varphi$, so let $A \in \varphi$ with $A \neq X$. Pick a set $B \in \varphi$ and a function $f \in \mathcal{F}$ with $f(B) = \{0\}$ and $f(X \setminus A) = \{1\}$. Since $B \subseteq X(f,r)$ for every $r > 0$, we have $f \in I$. Since $X(f, 1/2) \subseteq A$, the claim follows.

Suppose now that I is an ideal of $\mathcal{F}$. First, $X(f,r) \neq \emptyset$ for every $f \in I$ and for every $r > 0$ by Corollary 30. Next, let $f, g \in I$ and let $r > 0$. Since $|f|^2 + |g|^2 \in I$ and $X(|f|^2 + |g|^2, r) \subseteq X(f,r) \cap X(g,r)$, the set $\mathcal{B}(I)$ is a filter base on X. Since $\mathcal{B}(I)$ is an $\mathcal{F}$-family on X by Lemma 5, the filter φ on X generated by $\mathcal{B}(I)$ is an $\mathcal{F}$-filter. □

Theorem 33. Let I be an ideal of $\mathcal{F}$, let φ be the $\mathcal{F}$-filter on X generated by $\mathcal{B}(I)$, and let $f \in \mathcal{F}$. The following statements are equivalent:

(i) $f \in I$.
(ii) $\widehat{f}(p) = 0$ for every $p \in \overline{\varphi}$.
(iii) $X(f,r) \in \varphi$ for every $r > 0$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $f \in I$. Let $p \in \overline{\varphi}$ and let $r > 0$. Since φ is generated by $\mathcal{B}(I)$, we have $X(f,r) \in p$, and so $|\widehat{f}(p)| \leq r$ by Lemma 15. Therefore, $\widehat{f}(p) = 0$.

(ii) $\Rightarrow$ (iii) This follows from Lemma 22 and Theorem 11 (ii).

(iii) $\Rightarrow$ (i) Suppose that (iii) holds. We need only to show that $f \in \mathrm{cl}_{\mathcal{F}}(I)$, and so we may assume that $f \neq 0$. Let $0 < r < \|f\|$. Then $X(f,r) \neq X$. Since φ is generated by $\mathcal{B}(I)$, there exist some functions $h \in I$ and $g \in \mathcal{F}$ such that $g(X(h,1)) = \{0\}$ and $g(X \setminus X(f,r)) = \{1\}$. Now, $1/(|h| \vee 1)^2 \in \mathcal{F}$ by Lemma 29, so $k := |h|^2/(|h| \vee 1)^2 \in I$, and so $fk \in I$. Note that $k(X) \subseteq [0,1]$. Since $X(h,1) \subseteq X(f,r)$, the functions f and fk agree on $X \setminus X(f,r)$. Therefore, $\|f - fk\| = \sup_{x \in X(f,r)} |f(x)(1 - k(x))| \leq r$, and so $f \in \mathrm{cl}_{\mathcal{F}}(I)$, as required. □

Let I be an ideal of $\mathcal{F}$. The equalities $X(f,r) = X(|f|,r) = X(\overline{f},r)$ for every $f \in \mathcal{F}$ and for every $r > 0$ imply that $|f| \in I$ and $\overline{f} \in I$ for every $f \in I$.

Theorems 32 and 33 imply that, for every $\mathcal{F}$-filter φ on X, there exists a unique ideal I of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$. If I and J are ideals of $\mathcal{F}$ such that $I \subseteq J$, then $\mathcal{B}(I) \subseteq \mathcal{B}(J)$. From this we conclude the following: If I is an ideal of $\mathcal{F}$ and φ is the $\mathcal{F}$-filter on X generated by $\mathcal{B}(I)$, then I is a maximal ideal of $\mathcal{F}$ if and only if φ is an $\mathcal{F}$-ultrafilter on X. The following well-known property of $\mathcal{F}$ follows from Theorem 11 (ii).

Corollary 34. If I is an ideal of $\mathcal{F}$, then I is the intersection of all of those maximal ideals of $\mathcal{F}$ which contain I.

Remark 35. Let Δ be the spectrum of $\mathcal{F}$. We consider Δ as the space of all non-zero, multiplicative linear functionals on $\mathcal{F}$. The *evaluation mapping* $\varepsilon : X \to \Delta$ is defined by $[\varepsilon(x)](f) = f(x)$ for every $x \in X$ and for every $f \in \mathcal{F}$. If $x \in X$, then the $\mathcal{F}$-filter on X generated by $\mathcal{B}(\ker \varepsilon(x))$ is $\mathcal{N}_{\mathcal{F}}(x)$ by Lemma 13, By Theorem 25, the mapping $\mu \mapsto p_\mu$ from Δ to δX, where p_μ is the $\mathcal{F}$-ultrafilter on X generated by $\mathcal{B}(\ker \mu)$, is a homeomorphism.

8. $\mathcal{F}$-filters on topological spaces

In the previous sections, we made no assumption about any kind of structure on the set X. In this section, we assume that (X, τ) is a Hausdorff topological space and $\mathcal{F} \subseteq C(X)$.

Recall that A° denotes the $\tau(\mathcal{F})$-interior of a subset A of X. If $A \subseteq X$, then $e^{-1}(\widehat{A}) = A^\circ$. Since $\mathcal{F} \subseteq C(X)$, the set A° is open in X, and so the evaluation mapping $e : X \to \delta X$ is continuous. For every element $x \in X$, we denote by $\mathcal{N}(x)$ the neighborhood filter of x in (X, τ). Since $\mathcal{F} \subseteq C(X)$, we have $\mathcal{N}_{\mathcal{F}}(x) \subseteq \mathcal{N}(x)$ for every $x \in X$.

For the rest of this section, all the topological properties on X or on its subsets are taken with respect to the original topology τ of X.

Next theorem follows from Theorem 25.

Theorem 36. If Y is a compact Hausdorff space and $\varepsilon : X \to Y$ is a continuous mapping such that $\varepsilon(X)$ is dense in Y, then the following statements hold:

(i) The set $\mathcal{F} = \{h \circ \varepsilon : h \in C(Y)\}$ is a C^*-subalgebra of $C(X)$ containing the constant functions.
(ii) $\mathcal{F}$ is isometrically isomorphic to $C(Y)$.
(iii) There exists a homeomorphism $F : \delta X \to Y$ such that $F \circ e = \varepsilon$.

The evaluation mapping $e : X \to \delta X$ is an embedding if and only if the equality $\mathcal{N}(x) = \mathcal{N}_{\mathcal{F}}(x)$ holds for every $x \in X$. By Remark 5, the latter statement is equivalent to statement (ii) below.

Lemma 37. The following statements are equivalent:

(i) The canonical mapping $e : X \to \delta X$ is an embedding.
(ii) For every element $x \in X$ and for every neighborhood $A \in \mathcal{N}(x)$ with $A \neq X$, there exists a function $f \in \mathcal{F}$ with $f(x) = 1$ and $f(X \backslash A) = \{0\}$.

Statements (ii) and (iii) of the next corollary constitute the Stone-Weierstrass Theorem.

Corollary 38. Suppose that X is compact and that $\mathcal{F}'$ is a conjugate closed subalgebra of $C(X)$. Let $\mathcal{F}$ be the closure of $\mathcal{F}'$ in $C(X)$. The following statements are equivalent:

(i) The evaluation mapping $e : X \to \delta X$ is a homeomorphism.
(ii) $\mathcal{F}'$ separates the points of X.
(iii) $\mathcal{F} = C(X)$, that is, $\mathcal{F}'$ is dense in $C(X)$.

Proof. Since X is compact, the evaluation mapping $e : X \to \delta X$ is a continuous surjection. Therefore, e is a homeomorphism if and only if e is injective. Since $\mathcal{F}'$ is dense in $\mathcal{F}$, statements (i) and (ii) are equivalent.

(i) $\Rightarrow$ (iii) Suppose that $e : X \to \delta X$ is a homeomorphism. Then it is easy to verify that the mapping $g \mapsto g \circ e$ from $C(\delta X)$ to $C(X)$ is an isometric $*$-isomorphism. Since $\mathcal{F}$ is isometrically $*$-isomorphic with $C(\delta X)$ by Theorem 24, the statement follows.

(iii) $\Rightarrow$ (ii) This follows from Urysohn's Lemma. $\square$

Next statement is a consequence of the Gelfand-Naimark Theorem. Here, it follows from Remark 35 and Corollary 38.

Corollary 39. If X and Y are compact Hausdorff spaces, then X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are isometrically $*$-isomorphic.

We finish the paper with the following statement concerning locally compact spaces. If X is locally compact, we denote by X_∞ the one-point compactification of X. Let $e_1 : X \to X_\infty$ denote the natural embedding. Then it is easy to verify that $\{h \circ e_1 : h \in C(X_\infty)\} = C_0(X) \oplus \mathbb{C}$, where $\mathbb{C}$ denotes the constant functions on X.

Theorem 40. If X is non-compact and locally compact, then the following statements are equivalent:

(i) The evaluation mapping $e : X \to \delta X$ is an embedding and $e(X)$ is open in δX.
(ii) $C_0(X) \subseteq \mathcal{F}$.
(iii) There exists a continuous surjection $F : \delta X \to X_\infty$ such that $F(e(x)) = x$ for every $x \in X$.
(iv) The set $\varphi = \{X \setminus K : K \subseteq X \text{ and } \mathrm{cl}_X(K) \text{ is compact}\}$ is an $\mathcal{F}$-filter on X and $\mathcal{F}$ separates the points of X.

Proof. (i) $\Rightarrow$ (ii) Suppose that (i) holds. Let $f \in C_0(X)$. Define $F : \delta X \to \mathbb{C}$ by

$$F(p) = \begin{cases} f(x) & \text{if } p = e(x) \text{ for some } x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that F is continuous, and so $f \in \mathcal{F}$ by Corollary 24.

(ii) $\Rightarrow$ (iii) This follows from Theorems 26 and 36 and the remark preceding this theorem.

(iii) $\Rightarrow$ (iv) Since X is not compact, the set φ is a filter on X. Suppose now that (iii) holds. Clearly, $\mathcal{F}$ separates the points of X. To see that φ is an $\mathcal{F}$-filter on X, let $A \in \varphi$. Pick a subset K of X such that $A = X \setminus K$ and $\mathrm{cl}_X(K)$ is compact. Pick an open subset U of X such that $K \subseteq U$ and $\mathrm{cl}_X(U)$ is compact. By Urysohn's Lemma, there exists a function $h \in C(X_\infty)$ such that $h(\mathrm{cl}_X(K)) = \{1\}$ and $h(X_\infty \setminus U) = \{0\}$. By assumption and Corollary 24, there exists a function $f \in \mathcal{F}$ such that $\widehat{f} = h \circ F$. Since $F(e(x)) = x$ for every $x \in X$, we have $f(\mathrm{cl}_X(K)) = \{1\}$ and $f(X \setminus U) = \{0\}$. Since $X \setminus A \subseteq \mathrm{cl}_X(K)$ and $X \setminus U \in \varphi$, the statement follows.

(iv) $\Rightarrow$ (i) Suppose that (iv) holds. By Theorem 32, there exists an ideal I of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$. If $f \in I$, then $X(f, r) \in \varphi$ for every $r > 0$, and so we must have $f \in C_0(X)$. On the other hand, if $f \in C_0(X)$, then $X(f, r) \in \varphi$ for every $r > 0$. The proof of implication (iii) $\Rightarrow$ (i) in the proof of Theorem 33 applies to show that $f \in I$. Therefore, $I = C_0(X)$, and so the evaluation mapping is an embedding by Lemma 37 (see [2, p. 85]).

To finish the proof, it is enough to show that $\delta X \setminus e(X) = \widehat{\varphi}$. Since X is locally compact, the inclusion $\widehat{\varphi} \subseteq \delta X \setminus e(X)$ is obvious. To verify the reverse inclusion, let $p \in \delta X \setminus e(X)$ and let K be a compact subset of X. Since e is continuous, we have $p \in \delta X \setminus \overline{e(K)}$, and so $X \setminus K \in p$ by Lemma 16 (i). Therefore, $\varphi \subseteq p$, as required. $\square$

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